EIGENVALUES AND EIGENVECTORS OF THE EULER EQUATIONS IN GENERAL GEOMETRIES

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ABSTRACT

The complete eigensystem, including eigenvalues and left and right eigenvectors, of the Euler equations of inviscid flow are derived in a general finite volume coordinate frame. The symmetry of the eigenvector space is demonstrated from a mathematical and geometric viewpoint. Results are presented for 2-D and 3-D inviscid flow, and their application in computational fluid dynamics (CFD) is discussed.

INTRODUCTION

The eigensystem—eigenvalues and eigenvectors—of the Euler equations of inviscid flow form the basis of total variation diminishing (TVD) algorithms in computational fluid dynamics (CFD).[1-5] Whether the conservation equations are solved in a finite difference or finite volume format, the matrices of right and left eigenvectors that can be found in the literature are generally decomposed along the directions of a global (x, y, z) or local (ξ, η, ζ) coordinate system.[6-9] Such matrix decomposition, however, is not necessary. The eigenvalues and eigenvector matrix of 3-D inviscid flow can be expressed along any given direction, e.g. through a unit vector (n_x, n_y, n_z) normal to a surface. The resulting expression is relatively simple and allows for more efficient code implementation in finite volume TVD flow solvers.

GOVERNING EQUATIONS

The 3-D unsteady Euler equations of inviscid flow, a system of integral conservation equations for mass, momentum, and energy, can be written in vector notation as the sum of a volume and surface integral,

$$\frac{\partial}{\partial t}\int_{CV} \vec{Q}\, dV + \oint_{CS} \vec{F}\, dA = 0 \tag{1}$$

where,

$$\vec{Q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_o \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} \rho v_n \\ \rho u v_n + p n_x \\ \rho v v_n + p n_y \\ \rho w v_n + p n_z \\ \rho h_o v_n \end{bmatrix} \tag{2}$$

The velocity across a cell boundary is simply defined as the dot product of local velocity vector and outward unit normal vector to the boundary,

$$v_n = \vec{v}\cdot\hat{n} = u n_x + v n_y + w n_z$$
$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{3}$$

The stagnation energy and enthalpy per unit mass are the sum of static and dynamic parts, respectively,

$$e_o = e + e_k, \quad h_o = h + e_k$$
$$e_k = \tfrac{1}{2}\left(u^2 + v^2 + w^2\right) \tag{4}$$

with e_k being the kinetic energy per unit mass. Static energy, enthalpy, and pressure can all be expressed in terms of the local speed of sound a, a function of temperature, and the ratio of specific heats γ,

$$e = \frac{a^2}{\gamma(\gamma-1)}, \quad h = \frac{a^2}{\gamma-1}, \quad p = \frac{\rho a^2}{\gamma} \tag{5}$$

where,

$$a^2 = \gamma R T, \quad \gamma = c_p / c_v$$

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TRANSFORMATION MATRIX

The first step in determining the eigensystem of the above conservation equations is to derive the corresponding Jacobian or transformation matrix, which can be found by taking partial derivatives of the flux vector components F_i with respect to the flow vector components Q_j after expressing the flux vector solely in terms of the flow vector. Only the resulting transformation matrix shall be presented here,

$$[A]=\frac{\partial F_i}{\partial Q_j}=\begin{bmatrix} 0 & n_x & n_y & n_z & 0 \\ (\gamma-1)\,e_k\,n_x-u\,v_n & v_n-(\gamma-2)u\,n_x & u\,n_y-(\gamma-1)v\,n_x & u\,n_z-(\gamma-1)w\,n_x & (\gamma-1)n_x \\ (\gamma-1)\,e_k\,n_y-v\,v_n & v\,n_x-(\gamma-1)u\,n_y & v_n-(\gamma-2)v\,n_y & v\,n_z-(\gamma-1)w\,n_y & (\gamma-1)n_y \\ (\gamma-1)\,e_k\,n_z-w\,v_n & w\,n_x-(\gamma-1)u\,n_z & w\,n_y-(\gamma-1)v\,n_z & v_n-(\gamma-2)w\,n_z & (\gamma-1)n_z \\ \left[(\gamma-1)\,e_k-h_o\right]v_n & h_o\,n_x-(\gamma-1)u\,v_n & h_o\,n_y-(\gamma-1)v\,v_n & h_o\,n_z-(\gamma-1)w\,v_n & \gamma\,v_n \end{bmatrix} \tag{6}$$

We can now rewrite the Euler equations in the format of a general wave equation,

$$\frac{\partial}{\partial t}\int_{CV}\bar{Q}\,dV \;+\; \oint_{CS}\bar{F}(\bar{Q})\,dA \;=\; 0 \tag{7}$$

where,

$$\bar{F}(\bar{Q}) \;=\; [A]\,\bar{Q}$$

The transformation matrix [A] can be interpreted as a wave speed with local and directional dependence for a nonlinear multi-dimensional wave. The multi-dimensional character is really twofold: (1) we are working in a 3-D flow field, where waves can travel in any direction; (2) there are different types of waves, all traveling at their own characteristic speeds, which are determined by the eigenvalues of the matrix [A].

EIGENVALUES AND RIGHT EIGENVECTORS

The eigenvalues of the transformation matrix [A] are the roots λ_i of the characteristic equation,

$$\det\left(\,[A]-\lambda\,[I]\,\right) \;=\; 0 \tag{8}$$

where [I] is the identity matrix. It turns out that three eigenvalues are distinct and two are repeated,

$$\lambda_i \;=\; \{v_n-a\,,\; v_n\,,\; v_n+a\,,\; v_n\,,\; v_n\} \tag{9}$$

Each right eigenvector $\vec{R}_i$, corresponding to eigenvalue λ_i, must satisfy the following matrix equation,

$$[A]\,\vec{R}_i=\lambda_i\,\vec{R}_i \tag{10}$$

Being column vectors, the right eigenvectors can be collectively written in matrix form, such that,

$$\text{(R-1):}\qquad [R] \;=\; \left[\vec{R}_1\quad \vec{R}_2\quad \vec{R}_3\quad \vec{R}_4\quad \vec{R}_5\right] \;=\; \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ u-a\,n_x & u & u+a\,n_x & n_y & -n_z \\ v-a\,n_y & v & v+a\,n_y & -n_x & 0 \\ w-a\,n_z & w & w+a\,n_z & 0 & n_x \\ h_o-a\,v_n & e_k & h_o+a\,v_n & u\,n_y-v\,n_x & w\,n_x-u\,n_z \end{bmatrix} \tag{11}$$

It should be noted at this point that the eigenvectors of repeated eigenvalues are not distinct! They span a subspace and any vector within this subspace is also an eigenvector of the same repeated eigenvalue. In the general space $\mathbb{R}^5$ above—for which $\vec{R}_1$ through $\vec{R}_5$ form a basis—the eigenvectors $\vec{R}_4$ and $\vec{R}_5$, which belong to the repeated eigenvalues $\lambda_4=\lambda_5=v_n$, span a two dimensional subspace. Any linear combination of $\vec{R}_4$ and $\vec{R}_5$ is itself a member of that subspace and thus an eigenvector. For example, a sixth eigenvector, which would satisfy the equation $[A]\vec{R}_6 = \lambda_2 \vec{R}_6$, could be formed as follows,

$$\vec{R}_6 = \left(\frac{-n_z}{n_x}\right)\vec{R}_4 + \left(\frac{-n_y}{n_x}\right)\vec{R}_5 = \begin{bmatrix} 0 \\ 0 \\ n_z \\ -n_y \\ v\,n_z - w\,n_y \end{bmatrix} \qquad (12)$$

The following sets of right eigenvectors, again written in matrix format, are equally valid with (R-1),

$$\text{(R-2):} \qquad [R] = \begin{bmatrix} \vec{R}_1 & \vec{R}_2 & \vec{R}_3 & \vec{R}_4 & \vec{R}_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ u - a\,n_x & u & u + a\,n_x & n_y & 0 \\ v - a\,n_y & v & v + a\,n_y & -n_x & n_z \\ w - a\,n_z & w & w + a\,n_z & 0 & -n_y \\ h_o - a\,v_n & e_k & h_o + a\,v_n & u\,n_y - v\,n_x & v\,n_z - w\,n_y \end{bmatrix} \qquad (13)$$

$$\text{(R-3):} \qquad [R] = \begin{bmatrix} \vec{R}_1 & \vec{R}_2 & \vec{R}_3 & \vec{R}_5 & \vec{R}_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ u - a\,n_x & u & u + a\,n_x & -n_z & 0 \\ v - a\,n_y & v & v + a\,n_y & 0 & n_z \\ w - a\,n_z & w & w + a\,n_z & n_x & -n_y \\ h_o - a\,v_n & e_k & h_o + a\,v_n & w\,n_x - u\,n_z & v\,n_z - w\,n_y \end{bmatrix} \qquad (14)$$

LEFT EIGENVECTORS

The set of left eigenvectors can be determined from the inverse of the right eigenvector matrix,

$$[L] = [R]^{-1} \qquad (15)$$

For the first set of right eigenvectors (R-1), the matching set of left eigenvectors written in matrix format is,

$$\text{(L-1):} \qquad [L] = \begin{bmatrix} \overleftharpoon{L}_1 \\ \overleftharpoon{L}_2 \\ \overleftharpoon{L}_3 \\ \overleftharpoon{L}_4 \\ \overleftharpoon{L}_5 \end{bmatrix} = \begin{bmatrix} \frac{(\gamma-1)\,e_k + a\,v_n}{2\,a^2} & \frac{(1-\gamma)\,u - a\,n_x}{2\,a^2} & \frac{(1-\gamma)\,v - a\,n_y}{2\,a^2} & \frac{(1-\gamma)\,w - a\,n_z}{2\,a^2} & \frac{\gamma-1}{2\,a^2} \\ \frac{a^2 - (\gamma-1)\,e_k}{a^2} & \frac{(\gamma-1)\,u}{a^2} & \frac{(\gamma-1)\,v}{a^2} & \frac{(\gamma-1)\,w}{a^2} & \frac{1-\gamma}{a^2} \\ \frac{(\gamma-1)\,e_k - a\,v_n}{2\,a^2} & \frac{(1-\gamma)\,u + a\,n_x}{2\,a^2} & \frac{(1-\gamma)\,v + a\,n_y}{2\,a^2} & \frac{(1-\gamma)\,w + a\,n_z}{2\,a^2} & \frac{\gamma-1}{2\,a^2} \\ \frac{v - v_n\,n_y}{n_x} & n_y & \frac{n_y^2 - 1}{n_x} & \frac{n_y\,n_z}{n_x} & 0 \\ \frac{v_n\,n_z - w}{n_x} & -n_z & \frac{-n_y\,n_z}{n_x} & \frac{1 - n_z^2}{n_x} & 0 \end{bmatrix} \qquad (16)$$

Being row vectors, the left eigenvectors are denoted by a left pointing half arrow. Each left eigenvector and corresponding eigenvalue satisfy the following matrix equation, and thus bear the name "left" eigenvector,

$$\bar{L}_i\,[A] = \lambda_i\,\bar{L}_i \tag{17}$$

The above left eigenvector matrix becomes singular for $n_x = 0$, and simply multiplying the last two rows by n_x does not alleviate the problem; the matrix remains singular along certain directions, and a zero row vector emerges. It turns out that the inverse matrix of the second and third set of right eigenvectors, (R-2) and (R-3), yield a similar result, carrying the singularity in the n_y and n_z component, respectively,

(L-2):
$$[L] = \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \\ \bar{L}_3 \\ \bar{L}_6 \\ \bar{L}_7 \end{bmatrix} = \begin{bmatrix} \frac{(\gamma-1)\,e_k + a\,v_n}{2\,a^2} & \frac{(1-\gamma)\,u - a\,n_x}{2\,a^2} & \frac{(1-\gamma)\,v - a\,n_y}{2\,a^2} & \frac{(1-\gamma)\,w - a\,n_z}{2\,a^2} & \frac{\gamma-1}{2\,a^2} \\ \frac{a^2-(\gamma-1)\,e_k}{a^2} & \frac{(\gamma-1)\,u}{a^2} & \frac{(\gamma-1)\,v}{a^2} & \frac{(\gamma-1)\,w}{a^2} & \frac{1-\gamma}{a^2} \\ \frac{(\gamma-1)\,e_k - a\,v_n}{2\,a^2} & \frac{(1-\gamma)\,u + a\,n_x}{2\,a^2} & \frac{(1-\gamma)\,v + a\,n_y}{2\,a^2} & \frac{(1-\gamma)\,w + a\,n_z}{2\,a^2} & \frac{\gamma-1}{2\,a^2} \\ \frac{v_n\,n_x - u}{n_y} & \frac{1-n_x^2}{n_y} & -n_x & \frac{-n_x\,n_z}{n_y} & 0 \\ \frac{w - v_n\,n_z}{n_y} & \frac{n_x\,n_z}{n_y} & n_z & \frac{n_z^2-1}{n_y} & 0 \end{bmatrix} \tag{18}$$

(L-3):
$$[L] = \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \\ \bar{L}_3 \\ \bar{L}_8 \\ \bar{L}_9 \end{bmatrix} = \begin{bmatrix} \frac{(\gamma-1)\,e_k + a\,v_n}{2\,a^2} & \frac{(1-\gamma)\,u - a\,n_x}{2\,a^2} & \frac{(1-\gamma)\,v - a\,n_y}{2\,a^2} & \frac{(1-\gamma)\,w - a\,n_z}{2\,a^2} & \frac{\gamma-1}{2\,a^2} \\ \frac{a^2-(\gamma-1)\,e_k}{a^2} & \frac{(\gamma-1)\,u}{a^2} & \frac{(\gamma-1)\,v}{a^2} & \frac{(\gamma-1)\,w}{a^2} & \frac{1-\gamma}{a^2} \\ \frac{(\gamma-1)\,e_k - a\,v_n}{2\,a^2} & \frac{(1-\gamma)\,u + a\,n_x}{2\,a^2} & \frac{(1-\gamma)\,v + a\,n_y}{2\,a^2} & \frac{(1-\gamma)\,w + a\,n_z}{2\,a^2} & \frac{\gamma-1}{2\,a^2} \\ \frac{u - v_n\,n_x}{n_z} & \frac{n_x^2-1}{n_z} & \frac{n_x\,n_y}{n_z} & n_x & 0 \\ \frac{v_n\,n_y - v}{n_z} & \frac{-n_x\,n_y}{n_z} & \frac{1-n_y^2}{n_z} & -n_y & 0 \end{bmatrix} \tag{19}$$

Depending on the direction of the unit normal vector, a singularity in the left eigenvectors can thus be avoided by choosing the appropriate matrix. To minimize numerical error during the computation, the largest component of the normal vector—measured by its absolute value—should always be located in the denominator.

EIGENSYSTEM FOR 2-D FLOW

The transformation matrix and complete eigensystem for 2-dimensional flow can easily be derived from the more general 3-dimensional result by eliminating appropriate rows and columns within the matrices and simplifying the remainder by setting $w = n_z = 0$. For example, the transformation matrix $[A]$ for 2-D flow is obtained after eliminating the fourth row and fourth column, and by redefining some of the quantities involved,

$$\begin{gathered} v_n = \vec{v}\cdot\hat{n} = u\,n_x + v\,n_y \\ n_x^2 + n_y^2 = 1 \end{gathered} \tag{20}$$

$$e_k = \tfrac{1}{2}\left(u^2 + v^2\right) \tag{21}$$

The set of eigenvalues reduces to the first four, only one being repeated,

$$\lambda_i = \{v_n - a,\ v_n,\ v_n + a,\ v_n\} \tag{22}$$

Choosing the first set of eigenvectors, both left and right, the matrix of right eigenvectors for 2-D flow is obtained after eliminating the fourth row and fifth column from the general result, whereas the left eigenvector matrix is found by deleting the fifth row and fourth column from its original $5x5$ matrix. It is interesting to note that after applying the above 2-D definitions, the singularities in the last row of the new

4x4 left eigenvector matrix disappear! The first and third element of the fourth row can be simplified to,

$$\frac{v - v_n n_y}{n_x} = \frac{v - \left(u n_x + v n_y\right) n_y}{n_x} = \frac{v - u n_x n_y - v\left(1 - n_x^2\right)}{n_x} = v n_x - u n_y \tag{23}$$

$$\frac{n_y^2 - 1}{n_x} = \frac{-n_x^2}{n_x} = -n_x \tag{24}$$

DISCUSSION OF RESULTS

It was demonstrated earlier that two of the five right eigenvectors form a 2-dimensional subspace, within the general 5-dimensional space spanned by all right eigenvectors, and that every member of this subspace is itself an eigenvector. This phenomenon was attributed to the fact that their corresponding eigenvalues are repeated, which creates a "symmetry" within the eigenvector space. Although it may seem difficult to visualize any symmetry within a 5-dimensional vector space, part of this symmetry reveals itself when we geometrically interpret the 2-dimensional subspace as a plane. The eigenvectors $\vec{R}_4$, $\vec{R}_5$, and $\vec{R}_6$ shall now demonstrate this effect. Upon careful observation, they can be recast as shown in Equation (25), where $\vec{t}_x$, $\vec{t}_y$ and $\vec{t}_z$ are tangent vectors which all lie in the plane defined by the unit normal vector $\hat{n}$; their subscripts denote the vanishing component along the corresponding major axis, which can clearly be seen in Figure 1. Although all tangent vectors are depicted with equal length, they are not unit vectors and thus carry the standard vector symbol rather than the caret. Needless to say, the orthogonality relation holds between the tangent vectors and the surface unit normal, which is restated in Figure 1.

$$\vec{R}_4 = \begin{bmatrix} 0 \\ n_y \\ -n_x \\ 0 \\ u n_y - v n_x \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \vec{t}_z \\ \\ \vec{v} \cdot \vec{t}_z \end{bmatrix}$$

$$\vec{R}_5 = \begin{bmatrix} 0 \\ -n_z \\ 0 \\ n_x \\ w n_x - u n_z \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \vec{t}_y \\ \\ \vec{v} \cdot \vec{t}_y \end{bmatrix} \tag{25}$$

$$\vec{R}_6 = \begin{bmatrix} 0 \\ 0 \\ n_z \\ -n_y \\ v n_z - w n_y \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \vec{t}_x \\ \\ \vec{v} \cdot \vec{t}_x \end{bmatrix}$$

It was shown earlier that $\vec{R}_6$ can be expressed as a linear combination of $\vec{R}_4$ and $\vec{R}_5$. Geometrically, this implies that for each normal vector only two tangent vectors are needed to define the same plane. Any two tangent vectors rotated around the unit normal vector $\hat{n}$ will result in another set of equally valid tangent vectors defining the same planar surface. The unit normal vector, being the axis of rotation, can thus be seen as the axis of symmetry for the vector space defined.

$$\hat{n} \cdot \vec{t}_x = \hat{n} \cdot \vec{t}_y = \hat{n} \cdot \vec{t}_z = 0$$

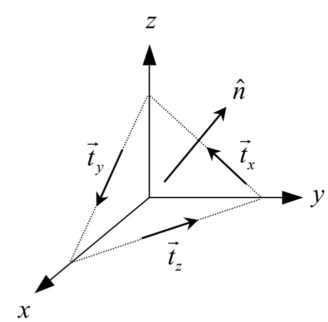

Figure 1: Eigenvector Subspace

What has been omitted so far is the physical interpretation of the eigenvectors themselves. In simple terms, which way do the eigenvectors point? It was mentioned earlier that the different speeds at which waves travel through the flow field are determined by their eigenvalues. The direction of wave travel has already been specified by the unit vector $\hat{n}$ normal to the surface under consideration—recall that we are trying to determine the magnitude of the mass, momentum, and energy flux across a given surface element.

The Euler equations contain three types, or families of waves, one for every distinct eigenvalue. Each family of waves carries a different *signal*. The waves traveling at the speed of the flow (v_n) are called entropy waves, their signal being entropy, whereas waves traveling at the speed of sound relative to the flow ($v_n \pm a$) are called acoustic waves. Unfortunately, the signal carried by acoustic waves is not quantifiable in simple thermodynamic terms, but let us just say that they carry acoustic information.

In essence, the eigenvectors point along the direction of the strongest signal. Any signal, whether physical or numerical in nature, is never completely noise free. Numerical noise is introduced into the flow field through discretization error, as well as the accumulative effect of machine round-off error. However, one can minimize the noise and thus obtain the strongest possible signal through proper *tuning*. The eigenvectors are optimally *tuned* with respect to the flow and thus deliver the best *signal-to-noise ratio* when it comes to computing the fluxes across a surface element.

CONCLUSION

The above derivation of the inviscid eigensystem was aided by the analysis software *Mathematica* from Wolfram Research, Inc. Although its presentation is by no means unique, the format adopted here is very compact and allows for efficient code implementation within a total variation diminishing (TVD) algorithm as part of an inviscid or viscous finite volume flow solver. An application of the above result can be found in the author's dissertation, which studies the high-altitude compressible viscous flow over a rotating disc.[10]

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